

Anomaly-free vector perturbations with holonomy corrections in loop quantum cosmology

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Abstract. We investigate vector perturbations with holonomy corrections in the framework of loop quantum cosmology. Conditions to achieve anomaly freedom for these perturbations are found at all orders. This requires the introduction of counter-terms in the hamiltonian constraint. We also show that anomaly freedom requires the diffeomorphism constraint to hold its classical form when scalar matter is added although the issue of a vector matter source, required for full consistency, remains to be investigated. The gauge-invariant variable and the corresponding equation of motion are derived. The propagation of vector modes through the bounce is finally discussed.

1. Introduction

In the canonical formulation of general relativity, the Hamiltonian is a sum of constraints. In particular, within the Ashtekar framework [1], the Hamiltonian is a sum of three constraints:

$$H_G[N^i, N^a, N] = \frac{1}{2\kappa} \int_{\Sigma} d^3x (N^i C_i + N^a C_a + NC) \approx 0, \quad (1)$$

where $\kappa = 8\pi G$, (N^i, N^a, N) are Lagrange multipliers, C_i is called the Gauss constraint, C_a is the diffeomorphism constraint, and C is the hamiltonian constraint. The sign " \approx " means equality on the surface of constraints (*i.e.* weak equality). One can also define the corresponding smeared constraints as follows:

$$\mathcal{C}_1 = G[N^i] = \frac{1}{2\kappa} \int_{\Sigma} d^3x N^i C_i, \quad (2)$$

$$\mathcal{C}_2 = D[N^a] = \frac{1}{2\kappa} \int_{\Sigma} d^3x N^a C_a, \quad (3)$$

$$\mathcal{C}_3 = S[N] = \frac{1}{2\kappa} \int_{\Sigma} d^3x NC, \quad (4)$$

that is such that $H_G[N^i, N^a, N] = G[N^i] + D[N^a] + S[N]$. The Hamiltonian is a total constraint which is vanishing for all multiplier functions (N^i, N^a, N) .

Because $H_G[N^i, N^a, N] \approx 0$ at all times, the time derivative of the Hamiltonian constraint is also weakly vanishing, $\dot{H}_G[N^i, N^a, N] \approx 0$. The Hamilton equation $\dot{f} = \{f, H_G[M^i, M^a, M]\}$ therefore leads to

$$\{H_G[N^i, N^a, N], H_G[M^i, M^a, M]\} \approx 0, \quad (5)$$

which, when explicitly written, means:

$$\{G[N^i] + D[N^a] + S[N], G[M^i] + D[M^a] + S[M]\} \approx 0. \quad (6)$$

Due to the linearity of the Poisson bracket, one can straightforwardly find that the condition (5) is fulfilled if the smeared constraints belong to a first class algebra

$$\{\mathcal{C}_I, \mathcal{C}_J\} = f^K_{IJ}(A_b^j, E_i^a) \mathcal{C}_K. \quad (7)$$

In (7), the $f^K_{IJ}(A_b^j, E_i^a)$ are structure functions which, in general, depend on the phase space (Ashtekar) variables (A_b^j, E_i^a) . The algebra of constraints is fulfilled at the classical level due to general covariance. To prevent the system from escaping the surface of constraints, leading to an unphysical behavior, the algebra must also be closed at the quantum level. In addition, it was pointed out in [2] that the algebra of quantum constraints should be strongly closed (*off shell* closure). This means that the relation (7) should hold in the whole kinematical phase space, and not only on the surface of constraints (*on shell* closure). This should remain true after promoting the constraints to quantum operators.

Loop quantum gravity (LQG) [3] is a promising approach to quantize gravity, based on a canonical formalism parametrized by Ashtekar variables. The methods of LQG applied to cosmological models are known as loop quantum cosmology (LQC)

[4]. In LQC, quantum gravity effects are introduced by holonomies of Ashtekar connection. This replacement is necessary because connection operators do not exist in LQG. Rewriting classical constraints in terms of holonomies leads to two types of quantum corrections: the so-called inverse-volume and holonomy corrections. Because the constraints are quantum-modified, the corresponding Poisson algebra might not be closed:

$$\{\mathcal{C}_I^Q, \mathcal{C}_J^Q\} = f^K{}_{IJ}(A_b^j, E_i^a) \mathcal{C}_K^Q + \mathcal{A}_{IJ}. \quad (8)$$

Here, \mathcal{A}_{IJ} stands for the anomaly term which can appear due to the quantum modifications. For consistency (closure of algebra), \mathcal{A}_{IJ} is required to vanish. The condition $\mathcal{A}_{IJ} = 0$ implies some restrictions on the form of the quantum corrections. In this paper, we will study this requirement to find a consistent way for introducing quantum holonomy corrections to the vector perturbations.

The question of the construction of an anomaly-free algebra of constraints is especially interesting to address in inhomogeneous LQC. Perturbations around the cosmological background are indeed responsible for structure formation in the Universe. This gives a chance to link quantum gravity effects with astronomical observations. In the particular case of the flat FLRW background, the Ashtekar variables can be decomposed as follows

$$A_a^i = \gamma \bar{k} \delta_a^i + \delta A_a^i \quad \text{and} \quad E_i^a = \bar{p} \delta_i^a + \delta E_i^a, \quad (9)$$

where \bar{k} and \bar{p} parametrize the background phase space, and γ is the so-called Barbero-Immirzi parameter.

The issue of anomaly freedom for the algebra of cosmological perturbations was extensively studied for inverse-volume corrections. It was shown that this requirement can be fulfilled for first order perturbations. This was derived for scalar [5, 6], vector [7] and tensor perturbations [8]. It is worth mentioning that, for the tensor perturbations, the anomaly-freedom is automatically satisfied. Based on the anomaly-free scalar perturbations, predictions for the power spectrum of cosmological perturbations were also performed [9]. This gave a chance to put constraints on some parameters of the model using observations of the cosmic microwave background radiation (CMB) [10].

The aim of this article is to address the issue of anomaly freedom for the holonomy-corrected vector perturbations in LQC. It was shown in [7] that these perturbations can be anomaly free up to the fourth order in the canonical variable \bar{k} . This, however, is not sufficient to perform the analysis of the propagation of vector modes through the cosmic bounce. Vector perturbations with *higher order holonomy corrections* were also recently studied [11]. It was shown there that, in this case, an anomaly-free formulation can be found for the gravitational sector. In this paper, we apply a different method, which is based on the introduction of counter-terms in the Hamiltonian constraint. We show that the anomaly freedom conditions for vector modes with holonomy corrections can be fulfilled in this way. The method is similar to the one used by Bojowald *et al.* in the case of cosmological perturbations with inverse-volume corrections. As we will see, the counter-terms do not introduce any higher-order holonomy corrections. This way of

fulfilling the anomaly freedom is therefore different from what was done in [11], where higher order terms are involved. Moreover, in Ref. [11], the issue of anomaly freedom was studied for the gravity sector only and the formulation suffers from ambiguities. In our study, scalar matter is introduced. The presence of this matter term fixes the ambiguity associated with the holonomy correction. It should be underlined that without a vector matter source, one cannot rigorously prove the anomaly cancellation. However, as will be shown in the next sections, our approach is meaningful as the equations derived are, as in [7], compatible with vector matter assuming $\pi_a = 0$ but $V_a \neq 0$.

Holonomy corrections arise while regularizing classical constraints, when expressing the Ashtekar connection in terms of holonomies. In particular, the regularization of the curvature of the Ashtekar connection F_{ab}^i leads to the factor $\left(\frac{\sin(\bar{\mu}\gamma\bar{k})}{\bar{\mu}\gamma}\right)^2$, which simplifies to \bar{k}^2 in the classical limit $\bar{\mu} \rightarrow 0$. However, the Ashtekar connection does not appear only because of F_{ab}^i : in the classical perturbed constraints, terms linear in \bar{k} are also involved. In principle, such terms should be holonomy-corrected. However, there is no direct expression for them, analogous to the regularization of the F_{ab}^i factor. Nevertheless, one can naturally expect that \bar{k} factors are corrected by the replacement[‡]

$$\bar{k} \rightarrow \frac{\sin(n\bar{\mu}\gamma\bar{k})}{n\bar{\mu}\gamma}, \quad (10)$$

where n is some unknown integer. It should be an integer because, when quantizing the theory, the $e^{i\gamma\bar{k}}$ factor is promoted to be the shift operator acting on the lattice states. If n was not an integer, the action of the operator corresponding to $e^{in\gamma\bar{k}}$ would be defined in a different basis. Another issue is related with the choice of $\bar{\mu}$, which corresponds to the so-called *lattice refinement*. Models with a power-law parametrization $\bar{\mu} \propto \bar{p}^\beta$ were discussed in details in the literature. While, in general, $\beta \in [-1/2, 0]$, it was pointed out that the choice $\beta = -1/2$ is favored [13]. This particular choice is called the $\bar{\mu}$ -scheme (new quantization scheme). Studies in this article are performed for the general power-law case $\bar{\mu} \propto \bar{p}^\beta$.

For the sake of simplicity, we use the notation

$$\mathbb{K}[n] := \begin{cases} \frac{\sin(n\bar{\mu}\gamma\bar{k})}{n\bar{\mu}\gamma} & \text{for } n \in \mathbb{Z}/\{0\}, \\ \bar{k} & \text{for } n = 0, \end{cases} \quad (11)$$

for the holonomy correction function. The introduction of holonomy corrections is therefore performed by replacing $\bar{k} \rightarrow \mathbb{K}[n]$. However, factors \bar{k}^2 are simply replaced by $\mathbb{K}[1]^2$, because they arise from the curvature of the Ashtekar connection. For the linear terms, the integers are parameters to be fixed.

[‡] this was derived rigorously e.g. for the Bianchi II model [12].

2. Vector perturbations with holonomy corrections

Vector modes within the canonical formulation were studied in [7]. It was shown there that

$$\delta E_i^a = -\bar{p}(c_1 \partial^a F_i + c_2 \partial_i F^a), \quad (12)$$

where $c_1 + c_2 = 1$ and the divergence-free condition $\delta_a^i \delta E_i^a = 0$ is fulfilled. The values of c_1 and c_2 depend on the gauge choice. However, due to the Gauss constraint, only symmetric variables are invariant under internal rotations. This is the case for $\delta E^{(a)}_i$, which is consequently independent on the specific choice of c_1 and c_2 , and should be preferred. The perturbation of the shift vector is parametrized as $\delta N^a = S^a$.

We consider the quantum holonomy-corrected hamiltonian constraint given by

$$S^Q[N] = \frac{1}{2\kappa} \int_{\Sigma} d^3x [\bar{N}(C^{(0)} + C^{(2)})], \quad (13)$$

where

$$C^{(0)} = -6\sqrt{\bar{p}} (\mathbb{K}[1])^2, \quad (14)$$

$$\begin{aligned} C^{(2)} = & -\frac{1}{2\bar{p}^{3/2}} (\mathbb{K}[1])^2 (1 + \alpha_1) (\delta E_j^c \delta E_k^d \delta_c^k \delta_d^j) + \sqrt{\bar{p}} (\delta K_c^j \delta K_d^k \delta_k^c \delta_j^d) \\ & - \frac{2}{\sqrt{\bar{p}}} (\mathbb{K}[v_1]) (1 + \alpha_2) (\delta E_j^c \delta K_c^j). \end{aligned} \quad (15)$$

Holonomy corrections were introduced by replacing $\bar{k} \rightarrow \mathbb{K}[n]$. Two counter-term functions α_1 and α_2 , whose interest will be made clear later, were also added. In the classical limit $\mathbb{K}[n] \rightarrow \bar{k}$, and $\alpha_i = \alpha_i(\bar{p}, \bar{k}) \rightarrow 0$, with $i = 1, 2$. We have assumed here that α_i are functions of the background variables only and that v_1 is an integer to be fixed. The hamiltonian constraint (13) corresponds to the one investigated in [7] while setting $\alpha_i = 0$. However, as we will show, it is necessary to introduce these additional factors, which vanish in the classical limit. These factors can, of course, also be viewed as contributions from the two counter-terms

$$S_{C1} = -\frac{\alpha_1}{2\kappa} \int_{\Sigma} d^3x \frac{\bar{N}}{2\bar{p}^{3/2}} (\mathbb{K}[1])^2 (\delta E_j^c \delta E_k^d \delta_c^k \delta_d^j), \quad (16)$$

$$S_{C2} = -\frac{\alpha_2}{2\kappa} \int_{\Sigma} d^3x \frac{2\bar{N}}{\sqrt{\bar{p}}} (\mathbb{K}[v_1]) (\delta E_j^c \delta K_c^j) \quad (17)$$

to the holonomy-corrected hamiltonian constraint.

A similar method of counter-terms was successfully applied for perturbations with inverse-volume corrections. In that case, it was possible to fix the counter-terms so as to make the algebra anomaly free. In this article, we follow the same path so as to find explicit expressions for α_1 and α_2 .

For the sake of completeness, we also introduce holonomy corrections to the diffeomorphism constraint, as follows:

$$D^Q[N^a] = \frac{1}{\kappa} \int_{\Sigma} d^3x \delta N^c [-\bar{p}(\partial_k \delta K_c^k) - (\mathbb{K}[v_2]) \delta_c^k (\partial_d \delta E_k^d)], \quad (18)$$

where v_2 is an unknown integer. It is worth emphasizing here that within LQG, the diffeomorphism constraint is fulfilled at the classical level while constructing the diffeomorphism invariant spin network states. If LQC was really derived from the full LQG theory, the classical form of the diffeomorphism constraint should therefore be used. However, at this early stage of the understanding of LQC, it might be safe to allow for some generalizations by introducing the holonomy correction also to the diffeomorphism constraint. This hypothesis was already studied in [14] in the case of holonomy-corrected scalar perturbations. It was assumed there that the holonomy correction function was given by $\mathbb{K}[2]$. In this work, we prefer to keep a more general expression $\mathbb{K}[v_2]$ with a free v_2 parameter. We will investigate whether this additional modification can help to fulfill the anomaly freedom conditions.

In order to investigate the algebra of constraints, the Poisson bracket has to be defined. We start with the gravity sector for which the Poisson bracket can be decomposed as follows:

$$\begin{aligned} \{\cdot, \cdot\} = & \frac{\kappa}{3V_0} \left(\frac{\partial \cdot}{\partial \bar{k}} \frac{\partial \cdot}{\partial \bar{p}} - \frac{\partial \cdot}{\partial \bar{p}} \frac{\partial \cdot}{\partial \bar{k}} \right) \\ & + \kappa \int_{\Sigma} d^3x \left(\frac{\delta \cdot}{\delta \delta K_a^i} \frac{\delta \cdot}{\delta \delta E_i^a} - \frac{\delta \cdot}{\delta \delta E_i^a} \frac{\delta \cdot}{\delta \delta K_a^i} \right). \end{aligned} \quad (19)$$

The algebra of constraints (13) and (18) shall now be investigated. Using the Poisson bracket (19), we find:

$$\{S^Q[N_1], S^Q[N_1]\} = 0, \quad (20)$$

$$\{D^Q[N_1^a], D^Q[N_2^a]\} = 0, \quad (21)$$

$$\begin{aligned} \{S^Q[N], D^Q[N^a]\} = & \frac{\bar{N}}{\sqrt{\bar{p}}} \mathcal{B} D^Q[N^a] \\ & + \frac{\bar{N}}{\kappa \sqrt{\bar{p}}} \int_{\Sigma} d^3x \delta N^c \delta_c^k (\partial_d \delta E_k^d) \delta E_k^d \mathcal{A}, \end{aligned} \quad (22)$$

where $\mathcal{B} := (1 + \alpha_2)\mathbb{K}[v_1] + \mathbb{K}[v_2] - 2\mathbb{K}[2]$, and \mathcal{A} is the anomaly function which, for reasons that shall be made clear later, is decomposed in two parts $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$, where

$$\mathcal{A}_1 = \mathcal{B}\mathbb{K}[v_2], \quad (23)$$

$$\begin{aligned} \mathcal{A}_2 = & 2\mathbb{K}[2]\bar{p} \frac{\partial \mathbb{K}[v_2]}{\partial \bar{p}} - \frac{1}{2}(\mathbb{K}[1])^2 \cos(v_2 \bar{\mu} \gamma \bar{k}) - 2\mathbb{K}[1]\bar{p} \frac{\partial \mathbb{K}[1]}{\partial \bar{p}} \cos(v_2 \bar{\mu} \gamma \bar{k}) \\ & + (1 + \alpha_2)\mathbb{K}[v_1]\mathbb{K}[v_2] - \frac{1}{2}\mathbb{K}[1]^2(1 + \alpha_1). \end{aligned} \quad (24)$$

This decomposition was made such that, in the classical limit ($\bar{\mu} \rightarrow 0$), both contributions to the anomaly vanish separately. Using the relation

$$\bar{p} \frac{\partial \mathbb{K}[n]}{\partial \bar{p}} = (\bar{k} \cos(n \bar{\mu} \gamma \bar{k}) - \mathbb{K}[n]) \beta, \quad (25)$$

the second contribution can be re-written as:

$$\begin{aligned} \mathcal{A}_2 = & -2\beta \mathbb{K}[2]\mathbb{K}[v_2] + (1 + \alpha_2)\mathbb{K}[v_1]\mathbb{K}[v_2] + (2\beta - 1/2)(\mathbb{K}[1])^2 \cos(v_2 \bar{\mu} \gamma \bar{k}) \\ & - \frac{1}{2}(\mathbb{K}[1])^2(1 + \alpha_1). \end{aligned} \quad (26)$$

where we have defined $x := \bar{\mu}\gamma\bar{k}$ and $x \in [0, \pi]$. Clearly, in the classical limit $\bar{\mu} \rightarrow 0$, the anomaly tends to zero. Requiring the anomaly cancellation up to the fourth order leads to the condition:

$$20 - 4v_1^2 - v_2^2 + 8\beta - 8v_2^2\beta = 0. \quad (30)$$

It can be shown that the condition of anomaly cancellation up to orders higher than four cannot be met. For $\beta = -1/2$ ($\bar{\mu}$ -scheme), the above equation simplifies to the quadratic Diophantine equation:

$$16 - 4v_1^2 + 3v_2^2 = 0. \quad (31)$$

This equation can be reduced to a Pell-type equation and solved for an infinite number of pairs of integers (v_1, v_2) . The first three solutions are $(2, 0)$, $(4, 4)$ and $(14, 16)$. The first one $(2, 0)$ corresponds to the case studied in [7], where the diffeomorphism constraint was kept at its classical form. The value $v_1 = 2$ obtained in this case was also used to fix the ambiguity for the holonomy-corrected tensor perturbations [8]. If the holonomy modified diffeomorphism constraint is used, the ambiguity cannot be fixed anymore due to the infinite number of solutions to Eq. (31).

As we have shown, the modification of the diffeomorphism constraint does not help satisfying the anomaly freedom conditions in the absence of counter-terms. In this case, the anomaly freedom can be fulfilled up to the fourth order in x . In the semi-classical limit $x \ll 1$, the anomaly cancellation up to the fourth order might be a good approximation. However, when approaching the bounce, where $x = \frac{\pi}{2}$, contributions from higher order terms become significant and the effects of the anomaly cannot be neglected anymore. Studies of vector perturbations during the bounce phase cannot be performed in such a setup. In order to study vector perturbations through the bounce, the anomaly cancellation at all orders is required. This probably makes mandatory the introduction of counter-terms.

3.2. The general case

Let us consider the general case with non-vanishing counter-terms. In this case, the requirement $\mathcal{A} = 0$ can be translated into a relation between the two counter-terms α_1 and α_2 :

$$\begin{aligned} \alpha_1 = & -1 + 4(1 + \alpha_2) \frac{\mathbb{K}[v_1]\mathbb{K}[v_2]}{\mathbb{K}[1]^2} - 4(1 + \beta) \frac{\mathbb{K}[2]\mathbb{K}[v_2]}{\mathbb{K}[1]^2} + 2 \frac{\mathbb{K}[v_2]^2}{\mathbb{K}[1]^2} \\ & + (4\beta - 1) \cos(v_2 \bar{\mu} \gamma \bar{k}). \end{aligned} \quad (32)$$

With this choice for the α_1 function, the anomaly is removed. However a significant ambiguity remains. Namely, the function α_2 together with parameters v_1 and v_2 remain undetermined. A particularly interesting case corresponds to the choice $\alpha_2 = 0$. This determines α_1 . Of course, this also works the other way round: one can set $\alpha_1 = 0$ and derive the correct expression for α_2 . Therefore, two special cases, heuristically

motivated, where one of the counter-terms is vanishing, are worth studying:

$$\alpha_1 = -1 + 4 \frac{\mathbb{K}[v_1]\mathbb{K}[v_2]}{\mathbb{K}[1]^2} - 4(1 + \beta) \frac{\mathbb{K}[2]\mathbb{K}[v_2]}{\mathbb{K}[1]^2} + 2 \frac{\mathbb{K}[v_2]^2}{\mathbb{K}[1]^2} + (4\beta - 1) \cos(v_2 \bar{\mu} \gamma \bar{k}), \quad (33)$$

$$\alpha_2 = 0, \quad (34)$$

and

$$\alpha_1 = 0, \quad (35)$$

$$\alpha_2 = -1 + \frac{1}{4} \frac{(\mathbb{K}[1])^2}{\mathbb{K}[v_1]\mathbb{K}[v_2]} + (1 + \beta) \frac{\mathbb{K}[2]}{\mathbb{K}[v_1]} - \frac{1}{2} \frac{\mathbb{K}[v_2]}{\mathbb{K}[v_1]} - (\beta - 1/4) \frac{(\mathbb{K}[1])^2 \cos(v_2 \bar{\mu} \gamma \bar{k})}{\mathbb{K}[v_1]\mathbb{K}[v_2]}. \quad (36)$$

To conclude, at least one counter-term is necessary to fulfill the anomaly freedom conditions for the gravity sector.

3.3. The $\mathcal{B} = 0$ case

Another possible way to fix the ambiguity in the choice of the α_1 and α_2 functions could be to set $\mathcal{B} = 0$. With this restriction, the anomaly cancellation is fulfilled by imposing $\mathcal{A}_2 = 0$ as $\mathcal{A}_1 \propto \mathcal{B} = 0$. As mentioned earlier, both \mathcal{A}_2 and \mathcal{A}_1 separately tend to zero in the classical limit, making this decomposition meaningful.

In this case, the Poisson bracket between the hamiltonian and diffeomorphism constraints is just $\{S^Q[N], D^Q[N^a]\} = 0$. The conditions $\mathcal{B} = 0$ and $\mathcal{A}_2 = 0$ can be translated into expressions for the α_1 and α_2 functions:

$$\alpha_1 = -1 + 4(1 - \beta) \frac{\mathbb{K}[2]\mathbb{K}[v_2]}{\mathbb{K}[1]^2} - 2 \frac{\mathbb{K}[v_2]^2}{\mathbb{K}[1]^2} + (4\beta - 1) \cos(v_2 \bar{\mu} \gamma \bar{k}), \quad (37)$$

$$\alpha_2 = -1 + \frac{2\mathbb{K}[2] - \mathbb{K}[v_2]}{\mathbb{K}[v_1]}. \quad (38)$$

The expressions for α_1 and α_2 are parametrized by the integers v_1 and v_2 only. However, the dependence upon v_1 vanishes when α_2 is used in the hamiltonian constraint.

The derived expressions for α_1 and α_2 do contain $\mathbb{K}[n]$ functions in the denominators. In principle, α_1 and α_2 could therefore diverge for some values of \bar{k} . However, in the counter-terms S_{C1} and S_{C2} , α_1 is multiplied by $\mathbb{K}[1]^2$ and α_2 by $\mathbb{K}[v_1]$. The subsequent cancellation prevents any physical divergence from occurring.

4. Introducing matter

We have shown that the gravity sector of the vector perturbations with holonomy corrections can be made anomaly free. We will now extend this result by introducing scalar matter. The matter Hamiltonian does not depend on the Ashtekar connection and

is therefore not subject to holonomy corrections. Furthermore, for vector perturbations, $\delta N = 0$. The matter Hamiltonian is perturbed up to the second order as follows:

$$H_m[N] = \bar{H}_m + \delta H_m = \int_{\Sigma} d^3x \bar{N} (C_m^{(0)} + C_m^{(2)}), \quad (39)$$

where

$$C_m^{(0)} = \bar{p}^{3/2} \left[\frac{1}{2} \frac{\bar{\pi}^2}{\bar{p}^3} + V(\bar{\varphi}) \right]. \quad (40)$$

The value of $C_m^{(2)}$ is given by

$$\begin{aligned} C_m^{(2)} = & \frac{1}{2} \frac{\delta \pi^2}{\bar{p}^{3/2}} + \frac{1}{2} \sqrt{\bar{p}} \delta^{ab} \partial_a \delta \varphi \partial_b \delta \varphi + \frac{1}{2} \bar{p}^{3/2} V_{,\varphi\varphi}(\bar{\varphi}) \delta \varphi^2 \\ & + \left(\frac{1}{2} \frac{\bar{\pi}^2}{\bar{p}^{3/2}} - \bar{p}^{3/2} V(\bar{\varphi}) \right) \frac{\delta_c^k \delta_d^j \delta E_j^c \delta E_k^d}{4 \bar{p}^2}, \end{aligned} \quad (41)$$

where we have used the condition $\delta_a^i \delta E_i^a = 0$. The matter diffeomorphism constraint is given by:

$$D_m[N^a] = \int_{\Sigma} d^3x \delta N^a \bar{\pi} (\partial_a \delta \varphi). \quad (42)$$

The total hamiltonian and diffeomorphism constraints are

$$S_{\text{tot}}[N] = S^Q[N] + H_m[N], \quad (43)$$

$$D_{\text{tot}}[N^a] = D^Q[N^a] + D_m[N^a]. \quad (44)$$

The resulting Poisson brackets are the following:

$$\{S_{\text{tot}}[N_1], S_{\text{tot}}[N_1]\} = 0, \quad (45)$$

$$\{D_{\text{tot}}[N_1^a], D_{\text{tot}}[N_2^a]\} = 0, \quad (46)$$

$$\begin{aligned} \{S_{\text{tot}}[N], D_{\text{tot}}[N^a]\} = & \frac{\bar{N}}{\sqrt{\bar{p}}} \mathcal{B} D^Q[N^a] + \frac{\bar{N}}{\kappa \sqrt{\bar{p}}} \int_{\Sigma} d^3x \delta N^c \delta_c^k (\partial_d \delta E_k^d) \delta E_k^d \mathcal{A} \\ & + [\cos(v_2 \bar{\mu} \gamma \bar{k}) - 1] \frac{\sqrt{\bar{p}}}{2} \left(\frac{\bar{\pi}^2}{2 \bar{p}^3} - V(\bar{\varphi}) \right) \int_{\Sigma} d^3x \bar{N} \partial_c (\delta N^a) \delta_a^j \delta E_j^c \\ & + \frac{\bar{\pi}}{\bar{p}^{3/2}} \int_{\Sigma} d^3x \bar{N} (\partial_a \delta N^a) \delta \pi - \bar{p}^{3/2} V_{\varphi}(\bar{\varphi}) \int_{\Sigma} d^3x \bar{N} (\partial_a \delta N^a) \delta \varphi. \end{aligned} \quad (47)$$

Anomaly freedom requires $\mathcal{B} = 0$, $\mathcal{A} = 0$, $v_2 = 0$ (classical diffeomorphism constraint), and also $\delta \varphi = 0 = \delta \pi$. The latter conditions $\delta \varphi = 0 = \delta \pi$ are due to the fact that metric scalar perturbations are not considered. Consistently, scalar field perturbations are vanishing too. In fact, one could set $\delta \varphi = 0 = \delta \pi$ from the very beginning but, without assuming this, it can be shown that the condition $\delta \varphi = 0 = \delta \pi$ in fact resulting from the anomaly freedom.

The associated counter-terms are given by (37) and (38) with $v_2 = 0$. Two non-vanishing counter-terms are required in contrast to the gravity sector, where only one counter-term was sufficient to fulfill the anomaly freedom conditions. The integer v_1 remains undetermined but the dependence upon this parameter cancels out in the

hamiltonian constraint. Namely, applying the counter-terms (37) and (38) with $v_2 = 0$, we find that the anomaly free hamiltonian constraint is given by:

$$S_{\text{free}}^Q[N] = \frac{1}{2\kappa} \int_{\Sigma} d^3x \left[\bar{N} (C_{\text{free}}^{(0)} + C_{\text{free}}^{(2)}) \right], \quad (48)$$

where

$$C_{\text{free}}^{(0)} = -6\sqrt{\bar{p}} (\mathbb{K}[1])^2, \quad (49)$$

$$\begin{aligned} C_{\text{free}}^{(2)} = & -\frac{1}{2\bar{p}^{3/2}} [4(1-\beta)\mathbb{K}[2]\bar{k} - 2\bar{k}^2 + (4\beta-1)\mathbb{K}[1]^2] (\delta E_j^c \delta E_k^d \delta_c^k \delta_d^j) \\ & + \sqrt{\bar{p}} (\delta K_c^j \delta K_d^k \delta_k^c \delta_j^d) - \frac{2}{\sqrt{\bar{p}}} (2\mathbb{K}[2] - \bar{k}) (\delta E_j^c \delta K_c^j). \end{aligned} \quad (50)$$

The gravitational diffeomorphism constraint holds its classical form ($v_2 = 0$). This is in agreement with LQG expectations. Interestingly, this can also be obtained here as a result of anomaly freedom.

The obtained anomaly-free Hamiltonian (48) is determined up to the choice of the $\bar{\mu}$ functions. There are no other remaining ambiguities. The $\bar{\mu}$ function appears in definition of the $\mathbb{K}[n]$ function. Because of this, there is also explicit appearance of the factor β in equation (50). The choice $\beta = -1/2$ is preferred by various considerations [13]. Recently, this value was shown to be required also by the conditions on the anomaly-free scalar perturbations with holonomy corrections [15]. For this choice of the β parameter, the remaining freedom is a parameter of proportionality in relation $\bar{\mu} \propto \bar{p}^{-1/2}$. This parameter can be written as $\sqrt{\Delta}$, so $\bar{\mu} = \sqrt{\Delta/\bar{p}}$. The parameter Δ has interpretation of physical area, around which the elementary holonomy is defined. It is expected that $\Delta \sim l_{Pl}^2$, where l_{Pl} is the Planck length. However, determination of the accurate value of Δ is a subject to empirical verifications.

It is worth noticing about the Hamiltonian constraint (48) that the effective holonomy corrections, due to the counter-terms, are no longer *almost periodic functions*, defined as follows [16]

$$f(\bar{k}) = \sum_n \xi_n e^{i\bar{\mu}\gamma\bar{k}n}. \quad (51)$$

In this expression, n runs over a finite number of integers and $\xi_n \in \mathbb{C}$. This does not lead to any problem at the classical level. However, difficulties may appear when going to the quantum theory on lattice states. This is because the quantum operator corresponding to \bar{k} does not exist in contrast to the $\mathbb{K}[n]$ functions, which are almost periodic functions. This problem does not exist if the gravitational sector, without any matter content, is considered alone. However, the diffeomorphism constraint then has to be holonomy corrected, as studied previously. In such a case, the background terms in the anomaly-free gravitational Hamiltonian are almost periodic functions. The loop quantization can therefore be directly performed.

5. Gauge invariant variable

The coordinate transformation $x^\mu \rightarrow x^\mu + \xi^\mu$ generates a tensor gauge transformation. In the case of vector modes, the coordinate transformation is parametrized by the shift vector $N^a = \xi^a$, where $\xi^a_{,a} = 0$. Therefore, the resulting gauge transformation is generated by the diffeomorphism constraint $\delta_\xi f = \{f, D^Q[\xi^a]\}$. The corresponding transformations for the canonical variables are:

$$\delta_\xi(\delta E_i^a) = \{\delta E_i^a, D^Q[\xi^a]\} = -\bar{p}\partial_i \xi^a, \quad (52)$$

$$\delta_\xi(\delta K_a^i) = \{\delta K_a^i, D^Q[\xi^a]\} = \mathbb{K}[v_2]\partial_a \xi^i. \quad (53)$$

Based on the equation of motion $\dot{E}_i^a = \{E_i^a, H_G\}$, and the definition (12), one finds the expression of δK_a^i . The dot means differentiation with respect to the conformal time since we have chosen $\bar{N} = \sqrt{\bar{p}}$. Using equations (52) and (53) one finds:

$$\delta_\xi F^a = \xi^a, \quad (54)$$

$$\delta_\xi S^a = \dot{\xi}^a + (2\mathbb{K}[2] - \mathbb{K}[v_1](1 + \alpha_2) - \mathbb{K}[v_2])\xi^a. \quad (55)$$

Based on this, one can define a gauge invariant variable

$$\sigma^a := S^a - \dot{F}^a - \underbrace{(2\mathbb{K}[2] - \mathbb{K}[v_1](1 + \alpha_2) - \mathbb{K}[v_2])}_{=-\mathcal{B}} F^a, \quad (56)$$

such that $\delta_\xi \sigma^a = 0$.

6. Equations of motion

In this section we derive the equation of motion for the gauge-invariant variable found in the previous section.

For the sake of completeness, we recall that the equations of motion for the background part are:

$$\dot{\bar{p}} = \bar{N}2\sqrt{\bar{p}}(\mathbb{K}[2]), \quad (57)$$

$$\begin{aligned} \dot{\bar{k}} = & -\frac{\bar{N}}{\sqrt{\bar{p}}} \left[\frac{1}{2}(\mathbb{K}[1])^2 + \bar{p}\frac{\partial}{\partial \bar{p}}(\mathbb{K}[1])^2 \right] \\ & + \frac{\kappa}{3V_0} \left(\frac{\partial \bar{H}_m}{\partial \bar{p}} \right), \end{aligned} \quad (58)$$

where $\bar{H}_m = V_0 \bar{N} C_m^{(0)}$ and $\bar{N} = \sqrt{\bar{p}}$. For a free scalar field, an analytical solution to these equations can be found [17]:

$$\bar{p} = \left(\frac{1}{6}\gamma^2 \Delta \pi_\varphi^2 \kappa + \frac{3}{2}\kappa \pi_\varphi^2 t^2 \right)^{1/3}. \quad (59)$$

This solution represents a symmetric bounce.

The diffeomorphism constraint $\frac{\delta}{\delta N^a} D_{\text{tot}}[N^a] = 0$ leads to the equation

$$\bar{p}(\partial_k \delta K_a^k) + (\mathbb{K}[v_2])\delta_a^k (\partial_d \delta E_k^d) = \kappa \bar{\pi} \partial_a (\delta \varphi). \quad (60)$$

Using the symmetrized variables

$$\begin{aligned}\delta K^{(i)}_{,a} &= \frac{1}{2} [(2\mathbb{K}[2] - \mathbb{K}[v_1](1 + \alpha_2)) (F_{a,}^i + F^i_{,a}) \\ &\quad + (F_{a,}^i + F^i_{,a}) \cdot - (S_{a,}^i + S^i_{,a})] \\ &= -\frac{1}{2} (\sigma_{a,}^i + \sigma^i_{,a}) + \frac{1}{2} \mathbb{K}[v_2] (F_{a,}^i + F^i_{,a}),\end{aligned}\tag{61}$$

and

$$\delta E^{(i)}_{,a} = -\bar{p} \frac{1}{2} (F_{a,}^i + F^i_{,a}),\tag{62}$$

equation (60) can be rewritten as

$$-\frac{\bar{p}}{2} \nabla^2 \sigma_a = \kappa \bar{\pi} \partial_a (\delta \varphi).\tag{63}$$

Because $\delta \varphi = 0$ (from the anomaly-free condition), the symmetric diffeomorphism constraint simplifies to the Laplace equation $\nabla^2 \sigma_a = 0$. Since, the spatial slice is flat ($\Sigma = \mathbb{R}^3$) there are no boundary conditions on σ_a . This restricts the possible solutions of the Laplace equation to $\sigma_a = b_a + d_a^c x_c$, where b_a and d_a^c are sets of constants. However, because σ_a is a perturbation (there is no contribution from the zero mode),

$$\int_{\Sigma} d^3x \sigma_a = 0,\tag{64}$$

as required from the consistency of the perturbative expansion. This is also the reason why the first order perturbation of the Hamiltonian is vanishing, $\int_{\Sigma} C^{(1)} d^3x = 0$. Condition (64) implies $b_a = 0$ and $d_a^c = 0$, which leads to $\sigma_a = 0$. This shows that our gauge invariant variable σ_a is identically equal to zero in absence of vector matter, in agreement with earlier studies [18]. This can also be proved by expanding σ_a into Fourier modes.

In order to have non-vanishing (physical) vector modes σ_a , a source term in equation (63) therefore has to be present. With "vector matter", this reads as [7]:

$$-\frac{1}{2\bar{p}} \nabla^2 \sigma_a = 8\pi G(\rho + P)V_a,\tag{65}$$

where ρ and P are the energy density and pressure of the vector matter and V_a is a matter perturbation vector. If $(\rho + P)V_a \neq 0$ then $\sigma_a \neq 0$ so physical vector perturbations are expected. However, it should be pointed out that proving that the formulation remains anomaly-free in presence of the vector matter remains an open issue. This could be checked, *e.g.*, by introducing an electromagnetic field in the Hamiltonian formulation [19]. We leave this problem to be analyzed elsewhere.

Due to the Gauss constraint, we introduce the symmetrized variable

$$\mathfrak{S}_a^i := \sigma^i_{,a} + \sigma_{a,}^i.\tag{66}$$

The equation of motion for this variable reads as:

$$-\frac{1}{2} \frac{d}{d\eta} \mathfrak{S}_a^i - \frac{1}{2} (2\mathbb{K}[2] + \mathcal{B}) \mathfrak{S}_a^i + \mathcal{A} F^i_{,a} = \kappa \bar{p} \delta T_a^{(i)},\tag{67}$$

where

$$\delta T_a^i = \frac{1}{\bar{p}} \left[\left(\frac{1}{3V_0} \frac{\partial \bar{H}_m}{\partial \bar{p}} \right) \left(\frac{\delta E_j^c \delta_a^j \delta_c^i}{\bar{p}} \right) + \frac{\delta H_m}{\delta \delta E_i^a} \right]. \quad (68)$$

For scalar matter $\delta T_a^i = 0$. The same holds for tensor modes [20] (the reasons are the same because $\delta_a^i \delta E_i^a = 0$ and $\delta N = 0$). When imposing the anomaly freedom conditions $\mathcal{A} = 0$ and $\mathcal{B} = 0$, equation (67) simplifies to

$$-\frac{1}{2} \frac{d}{d\eta} \mathfrak{S}_a^i - \frac{1}{2} \underbrace{(2\mathbb{K}[2])}_{=\frac{1}{\bar{p}} \frac{d\bar{p}}{d\eta}} \mathfrak{S}_a^i = 0, \quad (69)$$

with fully determined coefficients. Of course without vector matter, as discussed above, the variable \mathfrak{S}_a^i is equal to zero and the equation (69) is trivially satisfied. However, the presence on a non-vanishing contribution from V_a allows for non-trivial solutions of equation (69). In such a case, equation (69) leads to:

$$\mathfrak{S}_a^i = \frac{\text{const}}{\bar{p}} = \frac{\text{const}}{a^2}. \quad (70)$$

For a symmetric bounce driven by a free scalar field:

$$\mathfrak{S}_a^i \propto \frac{1}{\left(\frac{2\pi}{3\sqrt{3}} \gamma^3 l_{\text{Pl}}^2 + t^2 \right)^{1/3}}. \quad (71)$$

The evolution is smooth through the bounce. The amplitude of the perturbations grows during the contraction and decreases in the expanding phase. The maximum amplitude is reached at the transition point (bounce). Moreover, this evolution is independent on the length of the considered mode, as can be seen by performing a Fourier transform of the function σ_a . Because of this, there is significant difference with respect to tensor and scalar perturbations. For the scalar and tensor perturbations, the evolution is different depending on whether the mode length is shorter or longer than the Hubble horizon. In particular, on super-horizon scales, the amplitude of the scalar and tensor perturbations is *frozen*. In contrast, for the vector modes there is no such effect. Therefore, in an expanding universe, the amplitude of vector modes decreases with respect to the super-horizon tensor and scalar perturbations. The contribution from vector modes becomes negligible during the expansion phase. However, the situation reverses in the contracting phase, before the bounce. Then, the amplitude of the vector perturbations grows with respect to the super-horizon tensor and scalar perturbations. Therefore, on very large scales the vector perturbations can play an important role, *e.g.* leading to the generation of large scale magnetic fields [21]. This could lead to a new tool to explore physics of the (very) early universe.

7. Summary and conclusions

In this paper we have studied the issue of anomaly cancellation for vector modes with holonomy corrections in LQC. Our strategy is based on the introduction of counter-terms in the holonomy-corrected hamiltonian constraint. In our study, we have also

introduced possible holonomy corrections to the diffeomorphism constraint. We have shown, first, that the anomaly cancellation cannot be achieved without counter-terms. Holonomy corrections to the diffeomorphism constraint do not help significantly to fulfill the anomaly freedom conditions, that are anyway satisfied up to the fourth order in the canonical variable \bar{k} . Then, we have studied the anomaly issue for the gravitational sector with two counter terms. We have shown that the conditions of anomaly freedom can be met with at least one non-vanishing counter-term. The resulting effective holonomy corrections are almost periodic functions only if the diffeomorphism constraint is holonomy corrected. Subsequently, we have investigated the issue of anomaly cancellation when a matter scalar field is added. In this case, the closure conditions are more restrictive and fully determine the form of the resulting hamiltonian constraint. Moreover, this requires that the diffeomorphism constraint holds its classical form, in agreement with LQG expectations. Because of this, the effective holonomy corrections, which take into account contributions from the counter-terms, are no more almost periodic functions. We have found the gauge invariant variable and the corresponding equation of motion. The solution to this equation were also given. We have analyzed this solution for the symmetric bounce model to point out that the vector perturbations smoothly pass through the bounce, where their amplitude reaches its maximum but finite value. The work performed here for scalar matter should be extended to vector matter to fully address the considered issue.

In [15], we address the related issue of anomaly freedom for scalar perturbations with holonomy corrections. This is most important from the observational viewpoint.

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